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Optimal Stabilization of Takagi–Sugeno Fuzzy Systems with Application to Spacecraft Control

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A new design methodology is proposed for the optimal control of nonlinear systems described by the Takagi–Sugeno (TS) fuzzy model. The TS fuzzy systems are first classified into two families based on how diverse their input matrices are and then a controller synthesis procedure based on the inverse optimal approach is given for each family. We also show that the optimal controller can be found by solving a linear matrix inequality problem. The optimal controllers have robustness with respect to a class of input uncertainties. The proposed method is applied to the attitude control of a rigid spacecraft to demonstrate its validity.

Nomenclature

$A > 0$	=	symmetric matrix $A \in \Re^{n\times n}$ positive definite, $\boldsymbol{x}^T A \boldsymbol{x} > 0$ for any $\boldsymbol{x} \neq 0$
$B < 0$	=	$-B$ positive definite
$\mathrm{diag}[a_1, \ldots, a_n]$	=	diagonal matrix with order $a_1, \ldots, a_n$ along its diagonal
$I_n(0_n)$	=	identity (zero) matrix in $\Re^{n\times n}$
J	=	inertia matrix of the body
$L_f h(x)$	=	Lie derivative of a scalar function $h: \Re^n \to \Re^1$ with respect to a vector field $f: \Re^n \to \Re^n$, $\triangleq (\partial h/\partial x) f(x)$
$\Re^n$	=	normed linear space of real n vectors
$S(\cdot)$	=	skew-symmetric matrix
$\boldsymbol{u}$	=	acting control torque vector of the body, $[u_1\ \ u_2\ \ u_3]^T$
$\lambda_{\max}(C)$	=	maximum eigenvalue for a symmetric matrix $C \in \Re^{n\times n}$
ρ	=	Cayley–Rodrigues parameters vector (see Ref. 34) describing the body orientation, $[\rho_1\ \ \rho_2\ \ \rho_3]^T$
$\boldsymbol{\omega}$	=	angular velocity vector of the body in a body-fixed frame, $[\omega_1\ \ \omega_2\ \ \omega_3]^T$
$\|\cdot\|$	=	Euclidean norm; $\|x\|^2 = x^T x$, for $x \in \Re^n$

I. Introduction

SINCE Takagi and Sugeno[1] opened a new direction of research in the area of fuzzy control by introducing the Takagi–Sugeno (TS) fuzzy model, there have been several studies concerning the systematic design of stabilizing fuzzy controllers.[2–9] These studies have addressed the issue of stability for fuzzy control and have provided methodologies with rigorous stability proofs. Optimality is also an important concern in design of controllers. However, in the area of fuzzy control, it seems that how to design the optimal stabilizing controller has been seldom addressed.

In this paper, we propose a new design procedure yielding the optimal stabilizing controller for the nonlinear system described by a TS fuzzy model. In the TS fuzzy model,[1] the overall system is described by several fuzzy IF–THEN rules, each of which represents a local linear state equation $\dot{\boldsymbol{x}} = A_i \boldsymbol{x} + B_i \boldsymbol{u}$. To derive the optimal stabilizing controller, we employ the inverse optimal design approach of Sepulchre et al.[10] This approach was first proposed by Kalman[11] to establish the gain and phase margins of linear quadratic regulators and was recently revised by Freeman and Kokotović[12] to develop a design methodology of robust nonlinear controllers. The direct approach is based on seeking a controller that minimizes a given cost. The inverse optimal approach, however, avoids the task of solving a Hamilton–Jacobi–Bellman (HJB) equation but finds a stabilizing controller first and then shows that it is optimal with respect to a meaningful cost function.

For clear and convenient presentation of our results, we classify the TS fuzzy systems into two families based on how diverse the input matrix B_i is. Then, with a simple but clever choice of the optimal value function and the weight matrix, we propose an optimal controller synthesis method for each family. The resulting controllers are time-invariant state feedback or TS fuzzy controllers, depending on their input matrices. Also, we show that the parameters of the optimal stabilizing controller can be found by solving an linear matrix inequality (LMI) problem. The LMI formulation of the controller synthesis problems is of great practical value because it can be solved by using reliable and efficient convex optimization techniques,[13] for example, the LMI Control Toolbox of MATLAB®.[14]

To illustrate the synthesis procedure proposed in this paper, we apply the proposed method to the attitude control of a rigid spacecraft. The optimal control problem of a rigid body has been addressed by many researchers for the purpose of the control of spacecraft and aircraft.[15–19] Also, there have been several works that consider performance indices such as time and/or fuel in the formulation of the optimal control problems.[20–25] These studies have mainly addressed the optimal regulation problem for the angular velocity subsystem and for some quadratic costs.[20,26–28] Recently, the optimal attitude control problem of the complete system that includes the dynamics as well as the kinematics has been investigated by many researchers: Carrington and Junkins[29] have used a polynomial expansion approach to approximate the solution to the HJB equation. Rotea et al.[30] have shown that, for some special cases of performance outputs, Lyapunov functions that include a logarithmic term in the kinematic parameters result in linear controllers with a finite quadratic cost. For the general quadratic cost, they have also presented sufficient conditions that guarantee the existence of a linear, suboptimal, stabilizing controller. Tsiotras[31] has derived a new class of globally asymptotically stabilizing feedback control laws for the complete attitude motion of a nonsymmetric rigid body and

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has also presented a family of exponentially stabilizing optimal control laws for the complete system. Tsiotras,[32] by using the natural decomposition of the complete system into its kinematics and dynamics subsystems and the inherent passivity properties of these two subsystems, has presented a partial solution to the optimal regulation of the symmetry axis of a spinning rigid body. Bharadwaj et al.[33] have derived a couple of new globally stabilizing attitude control laws using the inverse optimal approach of Freeman and Kokotović,[12] where minimal, exponential coordinates are used to represent the kinematic equations.

In this paper, we consider the complete attitude motion of a rigid spacecraft described in terms of the Cayley–Rodrigues parameters (see Ref. 34) and observe that this system is, in fact, a system in cascade interconnection. To design a stabilizing control law for systems in this form, we can use the method of backstepping,[35] which was used by Sontag and Sussmann[36] for the first time to design feedback control laws for an underactuated rigid body. Tsiotras and Longuski[37] have employed this method for the attitude stabilization of an axisymmetric spacecraft with two control torques. In the present paper, we use the method of backstepping reported by Krstić and Tsiotras,[38] where a control Lyapunov function along with a stabilizing controller is derived and the stabilization problem is converted into a regulation problem.

Although the study of Krstić and Tsiotras[38] results in a very well-established optimal stabilization design for a rigid spacecraft, the design has an implicit assumption that we know the system parameters exactly. In many practical situations, however, this assumption may not be met. Thus, one may need an alternative design method to consider this practical issue together with the optimality in performance, which is the main motivation of our study.

The proposed method is based on the design of the optimal control law for the TS fuzzy model to handle uncertain system parameters and the optimality in performance. To the authors' best knowledge, the proposed approach is the first attempt to design the optimal stabilizing controller for a TS fuzzy system via the inverse optimal approach with application to stabilization of the complete attitude motion of a rigid spacecraft. A minor disadvantage of the proposed method is that it needs a computation procedure based on an LMI solver. However, the proposed method provides a simpler control law than that of Krstić and Tsiotras.[38]

The rest of this paper is organized as follows: First, preliminaries regarding TS fuzzy systems, quadratic stability, LMIs and design of optimal controllers via the inverse optimal approach are given. Next, based on the concept of the optimal stabilizing control, synthesis of the optimal controllers for the TS fuzzy systems is considered. Finally, for its verification, we apply the proposed method to the attitude control of a rigid spacecraft.

II. Preliminaries: TS Fuzzy Systems, Quadratic Stability and LMIs, and Inverse Optimal Design

A. TS Fuzzy Systems

The fuzzy model proposed by Takagi and Sugeno[1] consists of several fuzzy IF–THEN rules, each of which represents the local linear state equation of a nonlinear system. In this paper we consider a continuous TS fuzzy system described as follows.

Plant rule i:

IF $x_1(t)$ is M_{i1} and $\cdots$ and $x_n(t)$ is M_{in},

THEN

$$\dot{\boldsymbol{x}}(t) = A_i\boldsymbol{x}(t) + B_i\boldsymbol{u}(t), \qquad i = 1, \ldots, r \tag{1}$$

Here, $x_i(t), i = 1, \ldots, n$, and M_{ij}, $i = 1, \ldots, r$, $j = 1, \ldots, n$, are state variables and fuzzy sets, respectively, and r is the number of IF–THEN rules; $\boldsymbol{u}(t) \in \Re^p$ is the input vector and $A_i \in \Re^{n\times n}$ and $B_i \in \Re^{n\times p}$, $i = 1, \ldots, r$. Following the usual inference method of the TS fuzzy model, the state equation at time t is represented in the form of weighted average along the trajectory $\boldsymbol{x}(t) \in \Re^n$:

$$\dot{\boldsymbol{x}}(t) = \sum_{i=1}^{r} w_i[\boldsymbol{x}(t)]\{A_i\boldsymbol{x}(t) + B_i\boldsymbol{u}(t)\} \Bigg/ \sum_{i=1}^{r} w_i[\boldsymbol{x}(t)] \tag{2}$$

In Eq. (2), the weight functions are defined as

$$w_i[\boldsymbol{x}(t)] = \prod_{j=1}^{n} M_{ij}[x_j(t)]$$

where $M_{ij}[x_j(t)]$ is the grade of membership of $x_j(t)$ in the fuzzy set M_{ij}. The weight functions w_i, which are nonnegative and measurable, usually satisfy

$$\sum_{i=1}^{r} w_i[\boldsymbol{x}(t)] > 0, \qquad \text{for all } t > 0 \tag{3}$$

Throughout this paper, it is assumed that Eq. (3) always holds and that the vector $\boldsymbol{x}(t)$ can be measured in real time. With the normalization of weight functions

$$h_i[\boldsymbol{x}(t)] \triangleq w_i[\boldsymbol{x}(t)] \Bigg/ \sum_{i=1}^{r} w_i[\boldsymbol{x}(t)], \qquad i = 1, \ldots, r \tag{4}$$

the state equation (2) can be written in the polytopic form

$$\dot{\boldsymbol{x}}(t) = \sum_{i=1}^{r} h_i[\boldsymbol{x}(t)][A_i\boldsymbol{x}(t) + B_i\boldsymbol{u}(t)] \tag{5}$$

where the normalized weights h_i satisfy $h_i[\boldsymbol{x}(t)] \geq 0$, $i = 1, \ldots, r$, and

$$\sum_{i=1}^{r} h_i[\boldsymbol{x}(t)] = 1, \qquad \text{for all } t \geq 0$$

When the vector $\boldsymbol{x}(t)$ can be measured in real time, the TS fuzzy controller for the TS fuzzy model (1) is given by the following fuzzy IF–THEN implications.

Controller rule i:

IF $x_1(t)$ is M_{i1} and $\cdots$ and $x_n(t)$ is M_{in},

THEN

$$\boldsymbol{u}(t) = -K_i\boldsymbol{x}(t), \qquad i = 1, \ldots, r$$

Note that the TS fuzzy controller shares the same fuzzy set with the TS fuzzy model (1). The usual inference method yields the following representation for the TS fuzzy controller[4]:

$$\boldsymbol{u}(t) = -\sum_{i=1}^{r} h_i[\boldsymbol{x}(t)]K_i\boldsymbol{x}(t) \tag{6}$$

where the h_i are the same as that defined in Eq. (4). The parameters K_i of Eq. (6) should be chosen to meet the stability and performance requirements.

B. Quadratic Stability and LMIs

When $\boldsymbol{u}(t) = 0$ for all $t \geq 0$, the TS fuzzy system (5) becomes an input-free polytopic system given by

$$\dot{\boldsymbol{x}}(t) = \sum_{i=1}^{r} h_i[\boldsymbol{x}(t)]A_i\boldsymbol{x}(t) \tag{7}$$

As is well known from the stability theory, an autonomous dynamic system is stable if there exists a positive definite quadratic function $V(\boldsymbol{x}) = \boldsymbol{x}^T P\boldsymbol{x}$ that decreases along every nonzero trajectory of the system. A system having such a Lyapunov function is called quadratically stable. In the polytopic system (7), the time derivative of V along a nonzero trajectory $\boldsymbol{x}(\cdot)$ is given by

$$\frac{\mathrm{d}V}{\mathrm{d}t}(t) = \frac{\mathrm{d}}{\mathrm{d}t}[\boldsymbol{x}^T(t)P\boldsymbol{x}(t)] = \boldsymbol{x}^T(t)\left\{\sum_{i=1}^{r} h_i[\boldsymbol{x}(t)]A_i^T P + P\sum_{i=1}^{r} h_i[\boldsymbol{x}(t)]A_i\right\}\boldsymbol{x}(t) = \sum_{i=1}^{r} h_i[\boldsymbol{x}(t)]\boldsymbol{x}^T(t) \times \left\{A_i^T P + PA_i\right\}\boldsymbol{x}(t)$$

Then, we can see that the polytopic system (7) is quadratically stable if there exists a symmetric matrix P satisfying the following inequalities[2,13]:

$$P > 0, \qquad A_i^T P + PA_i < 0, \qquad i = 1, \ldots, r \tag{8}$$

Note that the left sides of these inequalities are all linear in the matrix variable P.

To find P satisfying Eq. (8) or to determine if there does not exist such P is a convex problem called the LMI feasibility problem. An LMI is any constraint of the form

$$A(\boldsymbol{x}) \triangleq A_0 + x_1 A_1 + \cdots + x_N A_N < 0 \tag{9}$$

where $\boldsymbol{x} \triangleq [x_1, \ldots, x_N]^T$ is the variable and $A_0, \ldots, A_N$ are given symmetric matrices. Since $A(\boldsymbol{y}) < 0$ and $A(z) < 0$ implies $A[(y + z)/2] < 0$, the LMI (9) is a convex constraint on the variable $\boldsymbol{x}$. It is well known that LMI-based optimization problems as well as LMI feasibility problems can be solved by interior-point algorithms with polynomial time,[13] and a toolbox of MATLAB[14] that is dedicated to convex problems involving LMIs is now available.

C. Inverse Optimal Design

We briefly review the inverse optimal design[10] for nonlinear control systems. One of the most important problems considered in the optimal control theory is to find a feedback control law $\boldsymbol{u}$ for the general nonlinear dynamic system

$$\dot{\boldsymbol{x}} = f(\boldsymbol{x}) + g(\boldsymbol{x})\boldsymbol{u} \tag{10}$$

with the following properties: 1) $\boldsymbol{u}$ achieves asymptotic stability of the equilibrium $\boldsymbol{x} = 0$ and 2) $\boldsymbol{u}$ minimizes the cost function

$$\mathcal{J} = \int_0^\infty [l(\boldsymbol{x}) + \boldsymbol{u}^T R(\boldsymbol{x})\boldsymbol{u}]\,\mathrm{d}t \tag{11}$$

where $l(\boldsymbol{x}) > 0$ and $R(\boldsymbol{x}) = R(\boldsymbol{x})^T > 0$ for all $\boldsymbol{x}$. When $\mathcal{J}$ is at its minimum, $\mathcal{J}(\boldsymbol{x})$ is called the optimal value function. As is shown in the next lemma,[10] this problem can be solved by considering the HJB equation.

Lemma 1: Suppose that there exists a positive definite function $V(\boldsymbol{x})$ that has continuous first partial derivatives with respect to $\boldsymbol{x}$ and that it satisfies the HJB equation

$$l(\boldsymbol{x}) + 4L_f V(\boldsymbol{x}) - 4[L_g V(\boldsymbol{x})]R^{-1}(\boldsymbol{x})[L_g V(\boldsymbol{x})]^T = 0$$
$$V(0) = 0, \tag{12}$$

and the feedback control $\boldsymbol{u} = -R^{-1}(\boldsymbol{x})[L_g V(\boldsymbol{x})]^T$, where $R(\boldsymbol{x}) = R(\boldsymbol{x})^T > 0$ for all $\boldsymbol{x}$, achieves asymptotic stability of the equilibrium point $\boldsymbol{x} = 0$ for system (10). Then the control law

$$\boldsymbol{u}^* = 2\boldsymbol{u} = -2R^{-1}(\boldsymbol{x})[L_g V(\boldsymbol{x})]^T$$

is the optimal stabilizing control law for system (10) that minimizes the cost function (11) over all $\boldsymbol{u}$ guaranteeing $\lim_{t\to\infty} \boldsymbol{x}(t) = 0$ and $4V(\boldsymbol{x})$ is the optimal value function.

To solve the HJB equation (12), in general, is not a feasible task. However, if the function $l(\boldsymbol{x})$ is a posteriori determined rather than a priori chosen by the designers, which is called the inverse optimal approach, one can solve the optimization problem more easily. Moreover, if we inspect the global properties of the optimality and stability, this is certainly the case when the optimal control $\boldsymbol{u}^*$ achieves global asymptotic stability of the equilibrium point $\boldsymbol{x} = 0$ for system (10), and the optimal value function $4V(\boldsymbol{x})$ is positive definite and radially unbounded. Thus, by the inverse optimal approach, which uses a positive definite and radially unbounded optimal value function, one can solve the optimization problem via the following lemma.[10]

Lemma 2: The control law $\boldsymbol{u}^*$ is an optimal, globally stabilizing control law for system (10) if 1) it achieves global asymptotic stability of $\boldsymbol{x} = 0$ for system (10) and 2) it is of the form

$$\boldsymbol{u}^* \triangleq 2\boldsymbol{u} = -2R^{-1}(\boldsymbol{x})[L_g V(\boldsymbol{x})]^T \tag{13}$$

where $R(\boldsymbol{x}) = R(\boldsymbol{x})^T > 0$ for all $\boldsymbol{x}$ and $V(\boldsymbol{x})$ is a radially unbounded, positive definite function such that

$$\dot{V}(\boldsymbol{x})|_{u=\frac{1}{2}u^*} \triangleq L_f V(\boldsymbol{x}) + \tfrac{1}{2}[L_g V(\boldsymbol{x})]\boldsymbol{u}^* < 0$$

Remark 1: One can derive Lemmas 1 and 2 from the arguments by Sepulchre et al.[10] by noting that the positive definite function $S(\boldsymbol{x}) = 4V(\boldsymbol{x})$ is a solution to the HJB equation (12). In this case, the optimal control law $\boldsymbol{u}^*$ of Eq. (13) is given by

$$\boldsymbol{u}^* \triangleq 2\boldsymbol{u} = -\tfrac{1}{2}R^{-1}(\boldsymbol{x})[L_g S(\boldsymbol{x})]^T$$

Note that we impose a positive definiteness condition to $l(\boldsymbol{x})$ in Eq. (11). This is obvious if we set $l(\boldsymbol{x}) := -4\dot{V}(\boldsymbol{x})|_{u=\frac{1}{2}u^*}$ and apply Lemmas 1 and 2.

III. Optimal Controller Synthesis for the TS Fuzzy System

In this section, we propose a synthesis procedure of the optimal controllers for nonlinear systems described by the TS fuzzy model. For the sake of clarity and convenience, we classify the TS fuzzy systems into two families based on how diverse their input matrices B_i are, and the controller synthesis procedure is given for each of these families.

A. Case 1: TS (B)

First, we consider the family of the TS fuzzy systems with a common input matrix:

$$B_1 = \cdots = B_r = B \tag{14}$$

We call this family TS (B). The state equation of the TS fuzzy systems in TS (B) can be described by

$$\dot{\boldsymbol{x}}(t) = \left\{ \sum_{i=1}^{r} h_i[\boldsymbol{x}(t)]A_i \boldsymbol{x}(t) \right\} + B\boldsymbol{u}(t) \tag{15}$$

With

$$f(\boldsymbol{x}) \triangleq \sum_{i=1}^{r} h_i(\boldsymbol{x})A_i\boldsymbol{x}$$

and $g(\boldsymbol{x}) \triangleq B$, Eq. (2) can be viewed as an example of a nonlinear system represented by the canonical form (10). Hence, Lemma 2 can be used to obtain the optimal, globally stabilizing controller $\boldsymbol{u}^*$ for the TS fuzzy system (15). Now, consider a radially unbounded, positive definite function $V(\boldsymbol{x})$ defined by $V(\boldsymbol{x}) = \boldsymbol{x}^T P \boldsymbol{x}$, where $P = P^T > 0$. If we set the weight function $R(\boldsymbol{x})$ to be the identity matrix, then the controller $\boldsymbol{u}^*$ of Eq. (13) can be reduced to

$$\boldsymbol{u}^* = -2[L_g V(\boldsymbol{x})]^T = -2\left[L_g(\boldsymbol{x}^T P \boldsymbol{x})\right]^T = -2[2\boldsymbol{x}^T P(B)]^T$$
$$= -4B^T P \boldsymbol{x} \triangleq -K\boldsymbol{x} \tag{16}$$

which is in the form of a time-invariant state feedback controller. With this controller $\boldsymbol{u}^*$ equation (16) applied to the TS fuzzy system (15), we have the closed-loop dynamics described by

$$\dot{\boldsymbol{x}}(t) = \left\{ \sum_{i=1}^{r} h_i[\boldsymbol{x}(t)]A_i - BK \right\} \boldsymbol{x}(t) \tag{17}$$

According to Lemma 2, the time-invariant state feedback controller $\boldsymbol{u}^*$ of equation (16) is qualified to be the optimal, globally stabilizing control law for the TS fuzzy system (15) if the following conditions hold:

$$P > 0$$

$$\dot{V}(\boldsymbol{x})|_{u=\frac{1}{2}u^*} = L_f V(\boldsymbol{x}) + \frac{1}{2}[L_g V(\boldsymbol{x})]\boldsymbol{u}^*$$
$$= \boldsymbol{x}^T \left[\sum_{i=1}^{r} h_i(\boldsymbol{x})\left(A_i^T P + P A_i\right) - 4PBB^T P \right] \boldsymbol{x} < 0 \tag{18}$$

Because the weight functions h_i satisfy

$$h_i(\boldsymbol{x}) \geq 0, \qquad i = 1, \ldots, r$$

and

$$\sum_{i=1}^{r} h_i(\boldsymbol{x}) = 1$$

the conditions (18) can be written as follows:

$$P > 0, \qquad A_i^T P + PA_i - 4PBB^T P < 0, \qquad i = 1, \ldots, r \tag{19}$$

By pre- and postmultiplying the second set of inequalities (19) by P^{-1}, and defining a variable $X \triangleq P^{-1}$, we obtain the synthesis procedure that yields the optimal, globally stabilizing controller for the TS fuzzy system (15).

The synthesis procedure for the family TS (B) is as follows:

1) Find $X = X^T \in \Re^{n\times n}$ satisfying

$$X > 0, \qquad A_i X + XA_i^T - 4BB^T < 0, \qquad i = 1, \ldots, r \tag{20}$$

2) Compute $P = X^{-1}$ and $K = 4B^T P$.

3) Set

$$\boldsymbol{u}^* = -K\boldsymbol{x} \tag{21}$$

B. Case 2: TS (B_i)

Next, we consider the family of the TS fuzzy systems whose input matrices are not same. We call this family TS (B_i), for which the state equation (2) can be written as

$$\dot{\boldsymbol{x}}(t) = \left\{\sum_{i=1}^{r} h_i[\boldsymbol{x}(t)]A_i\boldsymbol{x}(t)\right\} + \left\{\sum_{i=1}^{r} h_i[\boldsymbol{x}(t)]B_i\right\}\boldsymbol{u}(t) \tag{22}$$

With

$$f(\boldsymbol{x}) \triangleq \sum_{i=1}^{r} h_i(\boldsymbol{x})A_i\boldsymbol{x}, \qquad g(\boldsymbol{x}) \triangleq \sum_{i=1}^{r} h_i(\boldsymbol{x})B_i$$

Eq. (2) can be also viewed as another example of a nonlinear system represented by the canonical form (10). Hence, for the TS fuzzy system (22), we can utilize Lemma 2 to achieve the same result that we described in case 1. Observe from Eq. (13) that, with simple candidates chosen for $V(\boldsymbol{x})$ and $R(\boldsymbol{x})$, $\boldsymbol{u}^*$ can be reduced to a TS fuzzy controller.

More specifically, consider $V(\boldsymbol{x}) = \boldsymbol{x}^T P\boldsymbol{x}$, where $P = P^T > 0$. With $R(\boldsymbol{x}) = I$, the controller $\boldsymbol{u}^*$ of Eq. (13) can be reduced to

$$\begin{aligned}\boldsymbol{u}^* &= -2[L_g V(\boldsymbol{x})]^T = -2\big[L_g(\boldsymbol{x}^T P\boldsymbol{x})\big]^T \\ &= -2\left[2\boldsymbol{x}^T P\left(\sum_{i=1}^{r} h_i(\boldsymbol{x})B_i\right)\right]^T \\ &= -\sum_{i=1}^{r} h_i(\boldsymbol{x})\big(4B_i^T P\big)\boldsymbol{x} \triangleq -\sum_{i=1}^{r} h_i(\boldsymbol{x})K_i\boldsymbol{x}\end{aligned} \tag{23}$$

which is in the form of a TS fuzzy controller. With $\boldsymbol{u}^*$ of Eq. (23), the TS fuzzy system (22) has the closed-loop dynamics given by

$$\begin{aligned}\dot{\boldsymbol{x}}(t) &= \left\{\sum_{i=1}^{r}\sum_{j=1}^{r} h_i[\boldsymbol{x}(t)]h_j[\boldsymbol{x}(t)](A_i - B_iK_j)\right\}\boldsymbol{x}(t) \\ &= \left\{\sum_{i=1}^{r} h_i^2[\boldsymbol{x}(t)]Q_{ii} + 2\sum_{i<j}^{r} h_i[\boldsymbol{x}(t)]h_j[\boldsymbol{x}(t)] \right. \\ &\quad \left. \times \left(\frac{Q_{ij} + Q_{ji}}{2}\right)\right\}\boldsymbol{x}(t)\end{aligned} \tag{24}$$

where $Q_{ij} \triangleq A_i - B_i(4B_j^T P) = A_i - B_iK_j$. According to Lemma 2, the TS fuzzy controller $\boldsymbol{u}^*$ of equation (23) is qualified to be the optimal, globally stabilizing control law for the TS fuzzy system (22) if the following conditions hold:

$$P > 0$$

$$\begin{aligned}\dot{V}(\boldsymbol{x})|_{u=\frac{1}{2}u^*} &= L_f V(\boldsymbol{x}) + \frac{1}{2}[L_g V(\boldsymbol{x})]\boldsymbol{u}^* \\ &= \boldsymbol{x}^T\left[\sum_{i=1}^{r} h_i^2(\boldsymbol{x})\big(G_{ii}^T P + PG_{ii}\big) + 2\sum_{i<j}^{r} h_i(\boldsymbol{x})h_j(\boldsymbol{x}) \right. \\ &\quad \left. \times\left\{\left(\frac{G_{ij} + G_{ji}}{2}\right)^T P + P\left(\frac{G_{ij} + G_{ji}}{2}\right)\right\}\right]\boldsymbol{x} < 0\end{aligned} \tag{25}$$

where $G_{ij} \triangleq A_i - \frac{1}{2}B_i(4B_j^T P) = A_i - \frac{1}{2}B_iK_j$. Because the weight functions h_i satisfy

$$h_i(\boldsymbol{x})h_j(\boldsymbol{x}) \geq 0, \qquad i = 1, \ldots, r, \qquad j = 1, \ldots, r$$

$$\sum_{i=1}^{r}\sum_{j=1}^{r} h_i(\boldsymbol{x})h_j(\boldsymbol{x}) = 1$$

$$\sum_{i=1}^{r} h_i^2(\boldsymbol{x}) + 2\sum_{i<j}^{r} h_i(\boldsymbol{x})h_j(\boldsymbol{x}) = 1$$

the conditions (25) can be written as follows:

$$\begin{gathered}P > 0, \qquad G_{ii}^T P + PG_{ii} < 0, \qquad i = 1, \ldots, r \\ [(G_{ij} + G_{ji})/2]^T P + P[(G_{ij} + G_{ji})/2] < 0, \qquad 1 \leq i < j \leq r\end{gathered} \tag{26}$$

Define $X \triangleq P^{-1}$. With the same manner that is described in case 1, conditions (26) can then be transformed into the following stability criterion utilizing the vertices G_{ii} and $(G_{ij} + G_{ji})/2$ (Ref. 4):

$$\begin{gathered}X = P^{-1} > 0, \qquad G_{ii}X + XG_{ii}^T < 0, \qquad i = 1, \ldots, r \\ [(G_{ij} + G_{ji})/2]X + X[(G_{ij} + G_{ji})/2]^T < 0, \qquad 1 \leq i < j \leq r\end{gathered} \tag{27}$$

Hence, with conditions (27), we have the synthesis procedure that provides the optimal, globally stabilizing controller for the TS fuzzy system (22)

The synthesis procedure for the family TS (B_i) is as follows:

1) Find $X = X^T \in \Re^{n\times n}$ satisfying

$$\begin{gathered}X > 0, \qquad A_iX + XA_i^T - 4B_iB_i^T < 0, \qquad i = 1, \ldots, r \\ \tfrac{1}{2}A_iX + \tfrac{1}{2}XA_i^T + \tfrac{1}{2}A_jX + \tfrac{1}{2}XA_j^T - 2B_iB_j^T - 2B_jB_i^T < 0 \\ 1 \leq i < j \leq r\end{gathered} \tag{28}$$

2) Compute $P = X^{-1}$ and $K_i = 4B_i^T P$, $i = 1, \ldots, r$.

3) Set

$$\boldsymbol{u}^* = -\sum_{i=1}^{r} h_i(\boldsymbol{x})K_i\boldsymbol{x} \tag{29}$$

Remark 2: Note that the problems given by Eqs. (20) and (28) are LMI feasibility problems. One may use the function feasp of the LMI Control Toolbox,[14] which efficiently computes the solution of this problem. Following the syntax for this function, one can establish the numerical routines to solve the problems given by Eqs. (20) and (28) as follows: First, the variable $X = X^T \in \Re^{n\times n}$ is declared. Next, the LMIs (20) or (28) are specified, and the function feasp is declared. Finally, the function feasp computes the solution $X = X^T > 0$ for the given LMI problem.

Remark 3: An additional advantage of the optimal controllers (21) and (29) is that the closed-loop dynamics (17) and (24) have robustness with respect to a class of input uncertainties. An uncertainty included in such class is an unknown gain $k \in (\frac{1}{2}, \infty)$ or a static sector nonlinearity $\phi(\cdot) \in (\frac{1}{2}, \infty)$. This robustness property can be shown by the arguments by Sepulchre et al.[10]

Remark 4: In the optimal controller synthesis for the TS fuzzy system, we set the weight function $R(\boldsymbol{x}) = I$. Thus, by Lemmas 1 and 2, the proposed controllers are optimal with respect to the cost function

$$\mathcal{J} = \int_0^\infty [l(\boldsymbol{x}) + \boldsymbol{u}^{*T}\boldsymbol{u}^*]\,\mathrm{d}t$$

where

$$l(\boldsymbol{x}) := -4\dot{V}(\boldsymbol{x})|_{u=\frac{1}{2}u^*} = -4\left\{L_f V(\boldsymbol{x}) + \tfrac{1}{2}[L_g V(\boldsymbol{x})]\boldsymbol{u}^*\right\}$$
$$= -4L_f V(\boldsymbol{x}) + 4[L_g V(\boldsymbol{x})][L_g V(\boldsymbol{x})]^T$$

and

$$\boldsymbol{u}^* = -2[L_g V(\boldsymbol{x})]^T$$

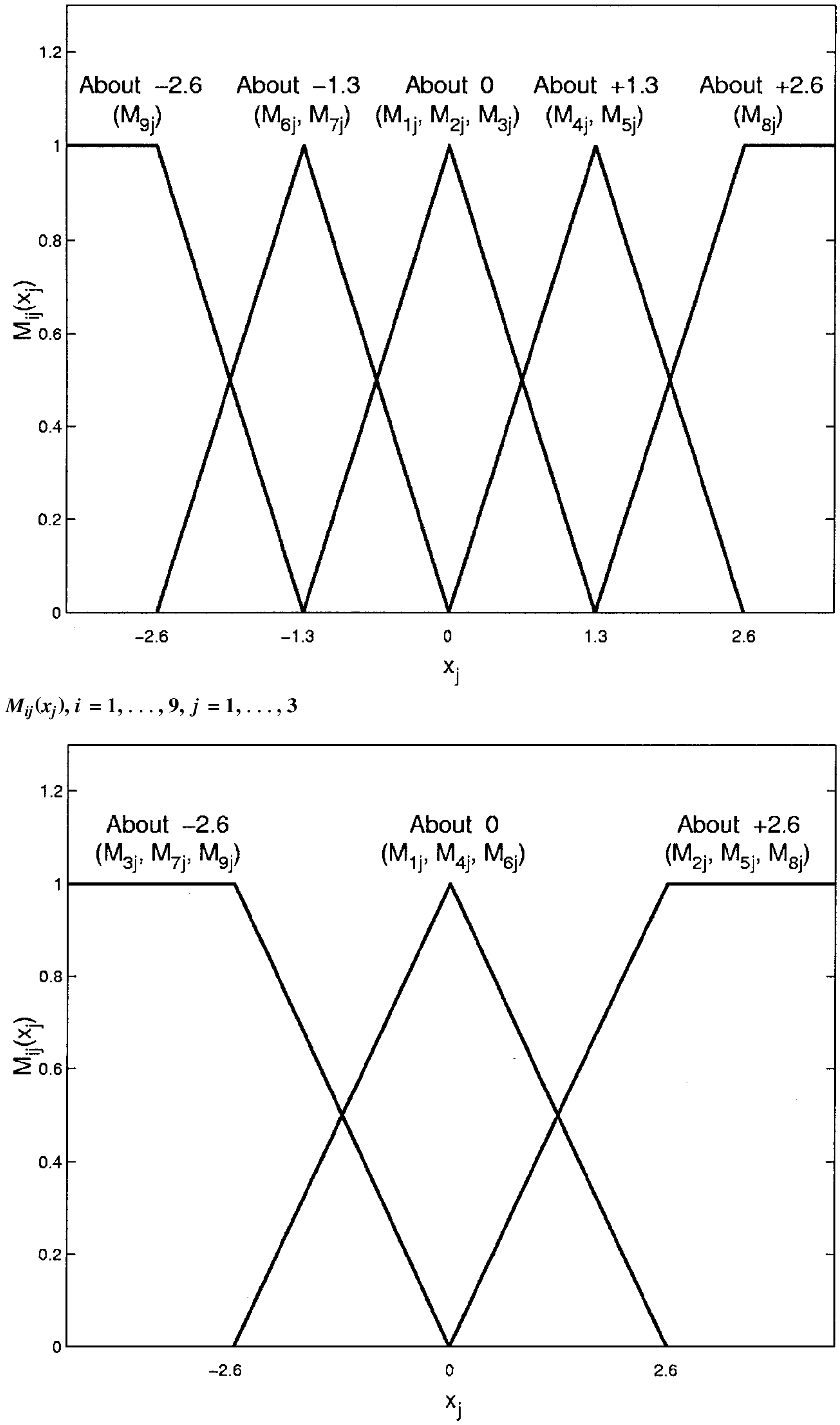
Note that we have $l(\boldsymbol{x}) > 0$ for all $\boldsymbol{x} \neq 0$ by Lemma 2. Consequently, the proposed method results in a cost function that imposes a positive penalty on the state and the control input for each $\boldsymbol{x}$.

IV. Numerical Example

In this section, we consider the attitude control of a rigid spacecraft. The complete attitude motion of a rigid spacecraft is given by the state equations[31,34]

$$\dot{\omega} = J^{-1}S(\omega)J\omega + J^{-1}\boldsymbol{u} \qquad (30a)$$

$$\dot{\rho} = H(\rho)\omega \qquad (30b)$$



$M_{ij}(x_j), i = 1, \ldots, 9, j = 1, \ldots, 3$

$M_{ij}(x_j), i = 1, \ldots, 9, j = 4, \ldots, 6$

Fig. 1 Membership functions of the fuzzy sets $M_{ij}(x_j)$.

where $J = J^T > 0$ and $S(\omega)$ is a 3×3 skew-symmetric matrix given by

$$S(\omega) := \begin{bmatrix} 0 & \omega_3 & -\omega_2 \\ -\omega_3 & 0 & \omega_1 \\ \omega_2 & -\omega_1 & 0 \end{bmatrix}$$

and the matrix-valued function $H\colon \Re^3 \to \Re^{3\times 3}$ denotes the kinematics Jacobian matrix given by

$$H(\rho) := \tfrac{1}{2}\left[I_3 - S(\rho) + \rho\rho^T\right]$$

We observe that the state equations (30a) and (30b) describe a system in cascade interconnection. To apply the proposed method to the cascade system of Eq. (30), we first use the method of backstepping reported by Krstić and Tsiotras[38] and convert the stabilization problem into a regulation problem. In the method of backstepping, we regard $\boldsymbol{\omega}$ as a virtual control input for the kinematics subsystem (30b), and the desired control law that stabilizes this subsystem has the form

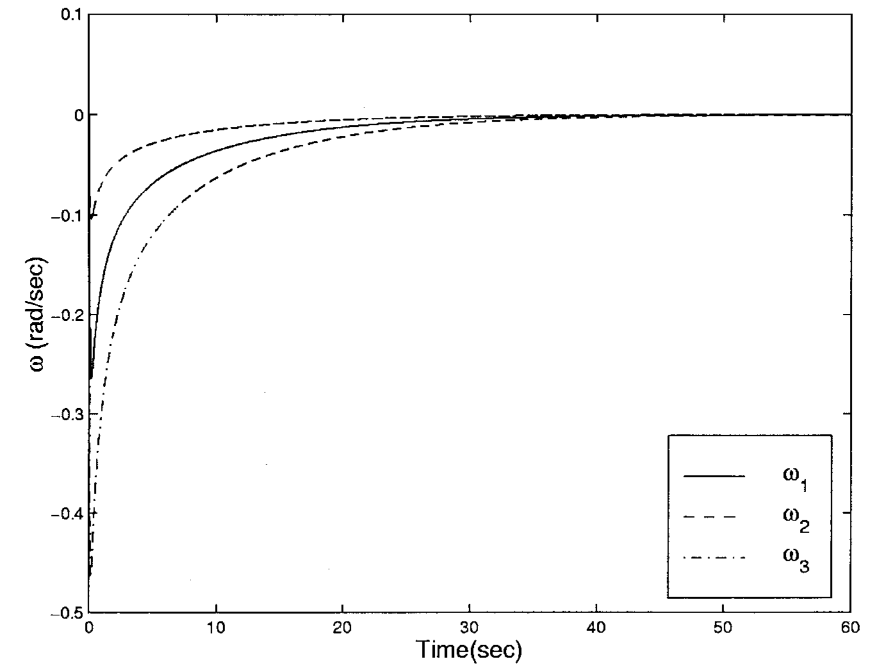
$$\omega_{\text{des}} = -k_1\rho, \qquad k_1 > 0 \tag{31}$$

Subsequently, if we design $\boldsymbol{u}$ to make $\boldsymbol{\omega}$ to follow $\boldsymbol{\omega}_{\text{des}}$ [Eq. (31)], then we can guarantee the stability of the subsystem (30a).[38] Define the error variable e as

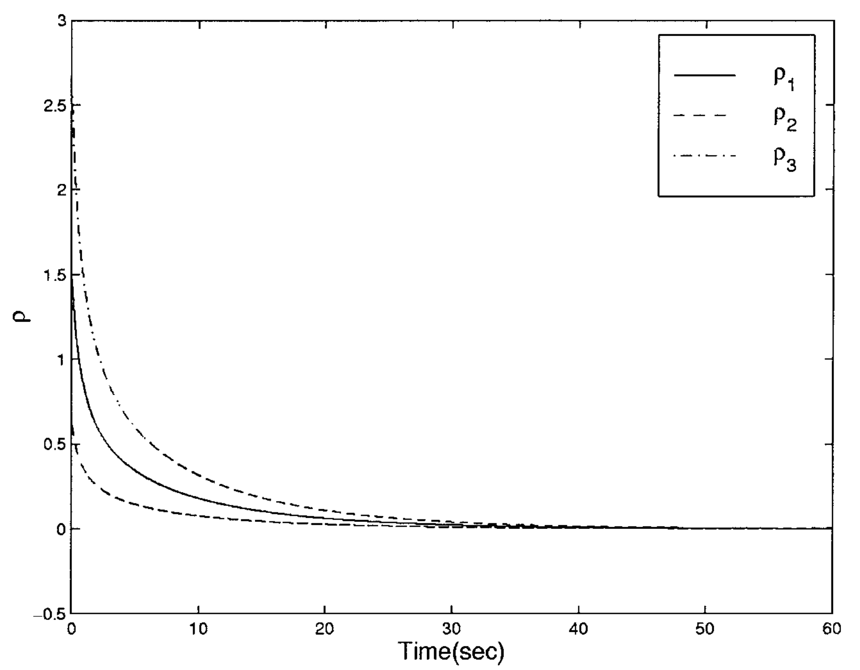
$$e = \omega - \omega_{\text{des}} = \omega + k_1\rho \tag{32}$$

Then, the differential equations for e [Eq. (32)] and ρ in the (e, ρ) coordinates are written as

$$\dot{e} = \left\{J^{-1}S(e - k_1\rho)J + k_1H(\rho)\right\}e - k_1\left\{J^{-1}S(e - k_1\rho)J + k_1H(\rho)\right\}\rho + J^{-1}u, \qquad \dot{\rho} = H(\rho)e - k_1H(\rho)\rho \tag{33}$$



Angular velocities response



Cayley–Rodrigues parameters response

Fig. 2 Responses of the system with the proposed controller $u^*(t)$.

Note that the stabilization problem of the complete system (30) can be converted into a regulation problem of the system (33). With $x_1 \triangleq e_1,\ x_2 \triangleq e_2,\ x_3 \triangleq e_3,\ x_4 \triangleq \rho_1,\ x_5 \triangleq \rho_2,\ x_6 \triangleq \rho_3,\ \boldsymbol{x}_e \triangleq [x_1\ x_2\ x_3]^T,\ \boldsymbol{x}_\rho \triangleq [x_4\ x_5\ x_6]^T$, and $\boldsymbol{x} \triangleq [\boldsymbol{x}_e^T\ \boldsymbol{x}_\rho^T]^T$, the system (33) can be represented by

$$\dot{\boldsymbol{x}} = A[\boldsymbol{x}(t)]\boldsymbol{x}(t) + Bu(t) \tag{34}$$

where $A(\boldsymbol{x})$ and B are

$$A(\boldsymbol{x}) \triangleq \begin{bmatrix} J^{-1}S(\boldsymbol{x}_e - k_1\boldsymbol{x}_\rho)J + k_1H(\boldsymbol{x}_\rho) & -k_1\left\{J^{-1}S(\boldsymbol{x}_e - k_1\boldsymbol{x}_\rho)J + k_1H(\boldsymbol{x}_\rho)\right\} \\ H(\boldsymbol{x}_\rho) & -k_1H(\boldsymbol{x}_\rho) \end{bmatrix}, \qquad B \triangleq \begin{bmatrix} J^{-1} \\ 0_3 \end{bmatrix}$$

For the numerical example, we chose $k_1 = 0.2$ and assume $J = \mathrm{diag}[10, 15, 20]$ (kg · m^2). Also, it is assumed that $\boldsymbol{x}_{ei}, \boldsymbol{x}_{\rho i} \in [-2.6, 2.6]$, $i = 1, \ldots, 3$. By sampling $A(\boldsymbol{x})$ at nine operating points of $[\boldsymbol{x}_{ei}\ \boldsymbol{x}_{\rho i}] = [0\ 0]$, $[0\ 2.6]$, $[0\ -2.6]$, $[1.3\ 0]$, $[1.3\ 2.6]$, $[-1.3\ 0]$, $[-1.3\ -2.6]$, $[2.6\ 2.6]$, and $[-2.6\ -2.6]$, $i = 1, \ldots, 3$, we can obtain the following TS fuzzy model for system (34).

Rule 1: IF x_1 is M_{11} (about 0) and x_2 is M_{12} (about 0) and x_3 is M_{13} (about 0) and x_4 is M_{14} (about 0) and x_5 is M_{15} (about 0) and x_6 is M_{16} (about 0) THEN $\dot{\boldsymbol{x}} = A_1\boldsymbol{x} + Bu$.

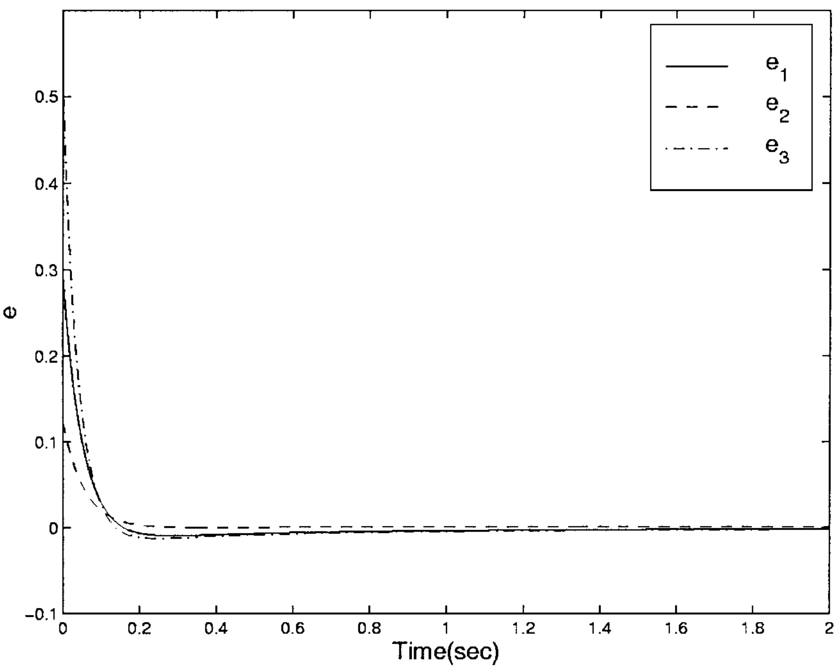


Fig. 3 Error variable *e* response using the proposed controller $u^*(t)$.

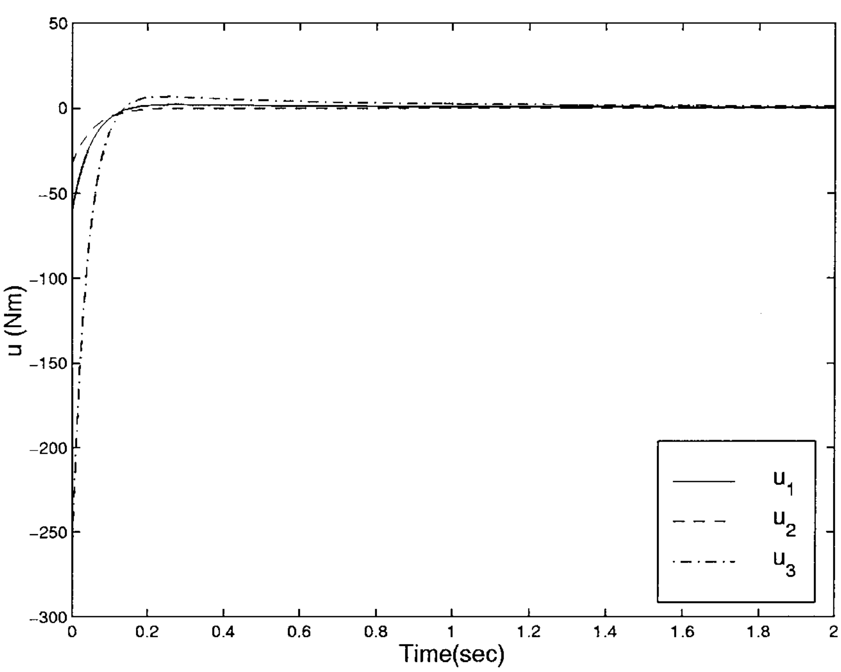


Fig. 4 Control inputs response using the proposed controller $u^*(t)$.

Rule 2: IF x_1 is M_{21} (about 0) and x_2 is M_{22} (about 0) and x_3 is M_{23} (about 0) and x_4 is M_{24} (about +2.6) and x_5 is M_{25} (about +2.6) and x_6 is M_{26} (about +2.6) THEN $\dot{\boldsymbol{x}} = A_2\boldsymbol{x} + Bu$.

Rule 3: IF x_1 is M_{31} (about 0) and x_2 is M_{32} (about 0) and x_3 is M_{33} (about 0) and x_4 is M_{34} (about −2.6) and x_5 is M_{35} (about −2.6) and x_6 is M_{36} (about −2.6) THEN $\dot{\boldsymbol{x}} = A_3\boldsymbol{x} + Bu$.

Rule 4: IF x_1 is M_{41} (about +1.3) and x_2 is M_{42} (about +1.3) and x_3 is M_{43} (about +1.3) x_4 is M_{44} (about 0) and x_5 is M_{45} (about 0) and x_6 is M_{46} (about 0) THEN $\dot{\boldsymbol{x}} = A_4\boldsymbol{x} + Bu$.

Rule 5: IF x_1 is M_{51} (about +1.3) and x_2 is M_{52} (about +1.3) and x_3 is M_{53} (about +1.3) and x_4 is M_{54} (about +2.6) and x_5 is M_{55} (about +2.6) and x_6 is M_{56} (about +2.6) THEN $\dot{\boldsymbol{x}} = A_5\boldsymbol{x} + Bu$.

Rule 6: IF x_1 is M_{61} (about −1.3) and x_2 is M_{62} (about −1.3) and x_3 is M_{63} (about −1.3) and x_4 is M_{64} (about 0) and x_5 is M_{65} (about 0) and x_6 is M_{66} (about 0) THEN $\dot{\boldsymbol{x}} = A_6\boldsymbol{x} + Bu$.

Rule 7: IF x_1 is M_{71} (about −1.3) and x_2 is M_{72} (about −1.3) and x_3 is M_{73} (about −1.3) and x_4 is M_{74} (about −2.6) and x_5 is M_{75} (about −2.6) and x_6 is M_{76} (about −2.6) THEN $\dot{\boldsymbol{x}} = A_7\boldsymbol{x} + Bu$.

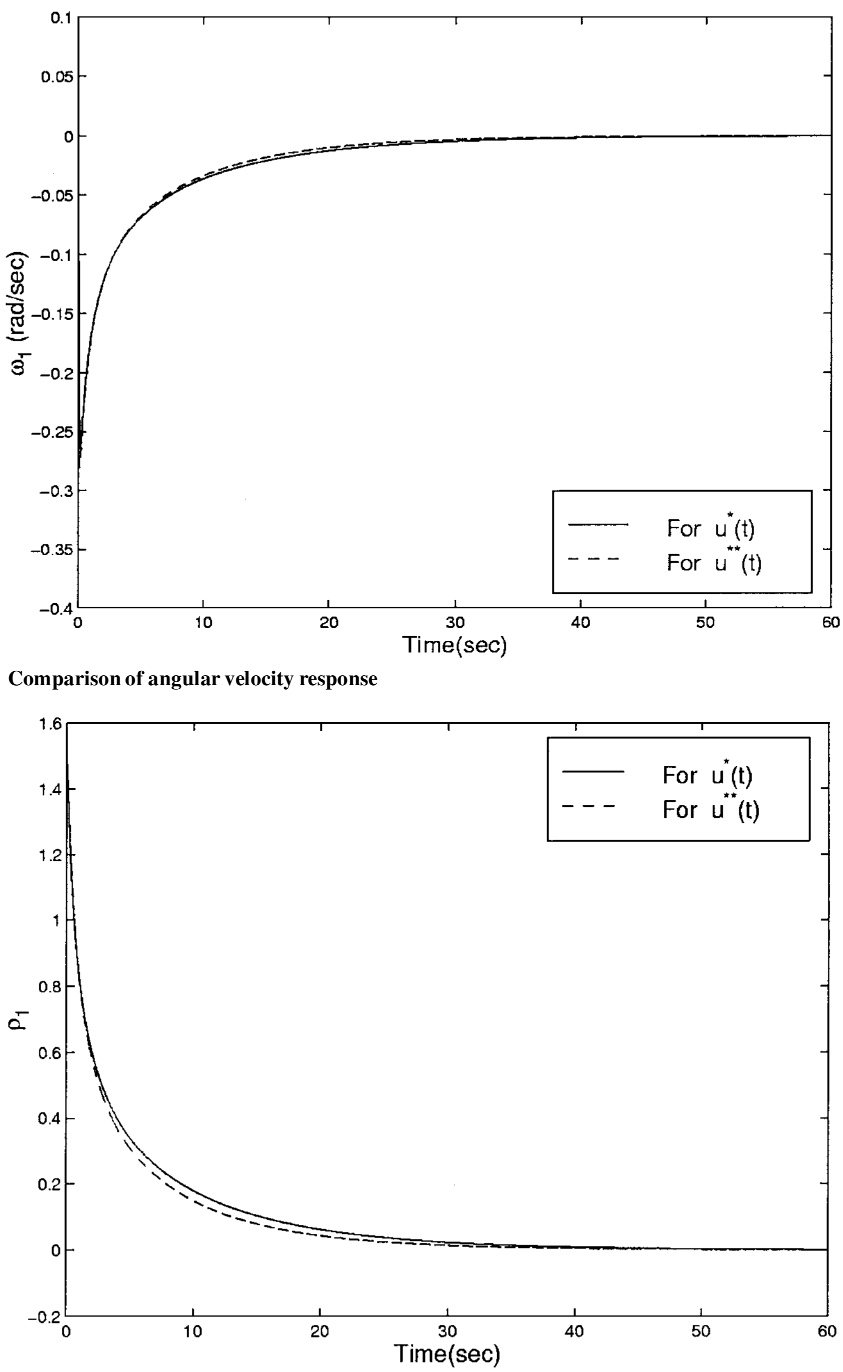
Rule 8: IF x_1 is M_{81} (about +2.6) and x_2 is M_{82} (about +2.6) and x_3 is M_{83} (about +2.6) and x_4 is M_{84} (about +2.6) and x_5 is M_{85} (about +2.6) and x_6 is M_{86} (about +2.6) THEN $\dot{\boldsymbol{x}} = A_8\boldsymbol{x} + Bu$.

Rule 9: IF x_1 is M_{91} (about −2.6) and x_2 is M_{92} (about −2.6) and x_3 is M_{93} (about −2.6) and x_4 is M_{94} (about −2.6) and x_5 is M_{95} (about −2.6) and x_6 is M_{96} (about −2.6) THEN $\dot{\boldsymbol{x}} = A_9\boldsymbol{x} + Bu$.

Here the state-space matrices A_i, which can be easily obtained by the substitution of each of the nine operating points to $A(\boldsymbol{x})$ with $k_1 = 0.2$, and B are given in the Appendix and the membership functions of the fuzzy sets M_{ij} are defined as in Fig. 1. With the normalized weights h_i defined by

$$h_i[\boldsymbol{x}(t)] \triangleq \prod_{j=1}^{6} M_{ij}[x_j(t)] \Bigg/ \sum_{i=1}^{9} \prod_{j=1}^{6} M_{ij}[x_j(t)], \qquad i = 1, \ldots, 9$$

the TS fuzzy model for system (34) can be transformed into the following polytopic form:



Comparison of angular velocity response

Comparison of Cayley–Rodrigues parameters response

Fig. 5 Controller comparison.

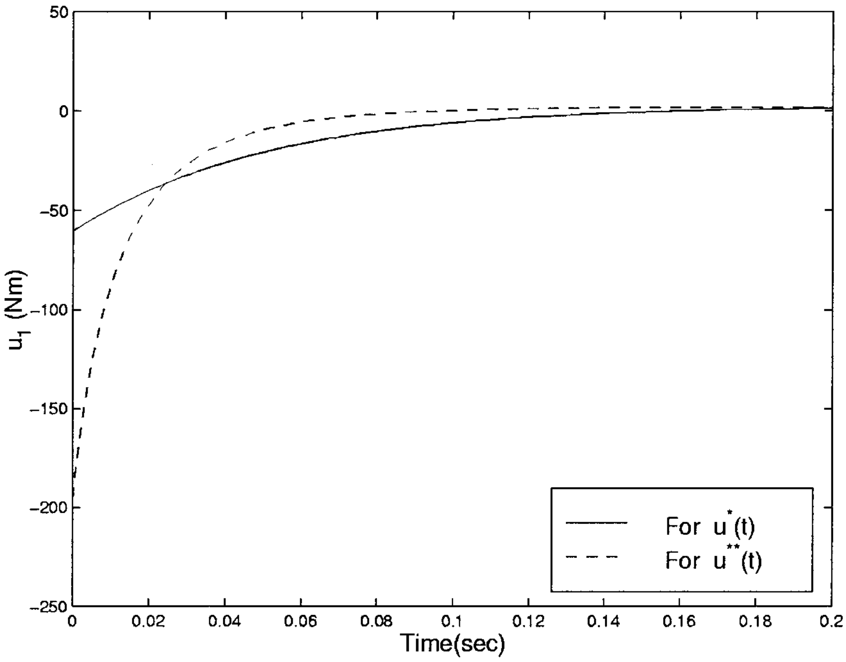


Fig. 6 Comparison of control input response for $u^*(t)$ and $u^{**}(t)$.

$$\dot{x}(t) = \left\{ \sum_{i=1}^{9} h_i[x(t)]A_i x(t) \right\} + Bu \tag{35}$$

In this fuzzy model, the dimensions of the state vector x and the input u are $n = 6$ and $p = 3$, respectively. Also, in the TS fuzzy system (35), $h_i[x(t)] \geq 0$ for all i and

$$\sum_{i=1}^{9} h_i[x(t)] = 1$$

Because this TS fuzzy model has the input matrix property common to Eq. (14), the control design procedure for TS (B) is readily applicable for the TS fuzzy system (35). Thus, with the diagonal matrix $P = P^T > 0$, we have

$$u(t) = -4B^T P x(t) \triangleq -Kx(t)$$

then the TS fuzzy system (35) has the closed-loop dynamics described by

$$\dot{x}(t) = \left\{ \sum_{i=1}^{9} h_i[x(t)]A_i - BK \right\} x(t) \tag{36}$$

Next, we illustrate the synthesis procedure for the family TS (B), in which the function feasp of the LMI Control Toolbox[14] is used to compute the solutions of LMIs.

Solving the LMIs (20) with the diagonal matrix $X = X^T > 0$ gives

$$X = \text{diag}[0.0020, 0.0010, 0.0004, 0.0112, 0.0110, 0.0105]$$

From $P = X^{-1}$ and $K = 4B^T P$, we obtain

$$P = \text{diag}[0.5112, 0.9935, 2.5712, 0.0895, 0.0913, 0.0957] \times 10^3$$

$$K = \left[\begin{array}{ccc|c} 204.4703 & 0 & 0 & \\ 0 & 264.9305 & 0 & 0_3 \\ 0 & 0 & 514.2326 & \end{array}\right] \triangleq [K_1 \mid 0_3]$$

Then the resulting optimal, globally stabilizing controller for the TS fuzzy system (35) is set to be

$$u^*(t) = -Kx(t) = -K_1 x_e(t) \tag{37}$$

Applying this controller (37) to the complete system (30) with $J = \text{diag}[10, 15, 20]$ $(\text{kg} \cdot \text{m}^2)$ we obtain the simulation results of Fig. 2 for the initial conditions $\omega(0) = [0\ 0\ 0]^T$ and $\rho(0) = [1.4735\ \ 0.6115\ \ 2.5521]^T$. The closed-loop stability is evident from these simulation results. Also, the trajectories of the error variable e [Eq. (32)] with $k_1 = 0.2$ and the corresponding control inputs $u^*(t)$ [Eq. (37)] are shown in Figs. 3 and 4. From Figs. 3 and 4, we observe that the initial control action substantially contributes to make $e \to 0$, that is, $\omega \to \omega_{\text{des}}$, within a short period of time.

To compare the performance between the proposed controller (37) and the controller proposed by Krstić and Tsiotras,[38] which is given by

$$u^{**}(t) = -\lambda_{\max}^2(J)\left[k_2 + \tfrac{3}{4}K_1 + (9/2k_1)\left(k_1^2\|\rho\|^2 + \|\omega + k_1\rho\|^2\right)\right]J^{-1}(\omega + k_1\rho) \tag{38}$$

we apply each of the proposed controller $u^*(t)$ [Eq. (37)] and the controller $u^{**}(t)$ [Eq. (38)] with $k_1 = 0.2$ and $k_2 = 0.1$ to the complete system (30) with $J = \text{diag}[10, 15, 20](\text{kg} \cdot \text{m}^2)$ for the same initial conditions. The simulation results are shown in Fig. 5, and the control inputs $u^*(t)$ of Eq. (37) and u^{**} of Eq. (38) are shown in Fig. 6. In Figs. 5 and 6, the solid lines represent the trajectories with the proposed controller (37) and the dashed lines represent the trajectories with the controller (38). The comparisons with the controller (38) of Krstić and Tsiotras[38] show that the proposed controller yields almost the same convergence rate to the equilibrium state as the controller (38), but with a smaller control effort.

The merit of the proposed method is that it does not require the exact system parameters. This is due to the fuzzy modeling procedure. In this procedure, we represent the system as the set of linear approximations to incorporate linguistic descriptions in the form of IF–THEN rules and obtain the TS fuzzy system by the fuzzy blending. Thus, the TS fuzzy system is a nonlinear system that approximates the system to be controlled. Then, we design the optimal controller for the TS fuzzy system. On the other hand, the design of Krstić and Tsiotras[38] is based on the assumption that we know the system parameters exactly. In practice, however, this assumption may not be met and the controller may not have sufficient robustness to parameter uncertainties in the plant dynamics. The detailed discussions on this problem can be found by Keel and Bhattacharyya.[39]

V. Conclusions

In this paper, we propose a new design methodology for the optimal control of nonlinear systems described by the TS fuzzy model. The TS fuzzy systems are classified into two families based on how diverse their input matrices are, and a controller synthesis is proposed for each family. The derivation of the optimal controllers makes use of the inverse optimal control theory, and the optimal controllers have robustness with respect to a class of input uncertainties. The attitude control of a spacecraft is then considered to illustrate the proposed method. The design procedure is essentially based on the LMI feasibility problem and solved by using MATLAB to result in satisfactory simulation results. Further investigations may consider the refinement of the proposed procedure by incorporating other performance requirements such as decay rate and input bound.

Appendix: State Space Matrices A_i and B

$A_1 =$

$$\begin{bmatrix} 0.1000 & 0.0000 & 0.0000 & -0.0200 & 0.0000 & 0.0000 \\ 0.0000 & 0.1000 & 0.0000 & 0.0000 & -0.0200 & 0.0000 \\ 0.0000 & 0.0000 & 0.1000 & 0.0000 & 0.0000 & -0.0200 \\ 0.5000 & 0.0000 & 0.0000 & -0.1000 & 0.0000 & 0.0000 \\ 0.0000 & 0.5000 & 0.0000 & 0.0000 & -0.1000 & 0.0000 \\ 0.0000 & 0.0000 & 0.5000 & 0.0000 & 0.0000 & -0.1000 \end{bmatrix}$$

$A_2 =$

$$\begin{bmatrix} 0.7760 & -0.3640 & 1.9760 & -0.1552 & 0.0728 & -0.3952 \\ 1.2827 & 0.7760 & -0.2773 & -0.2565 & -0.1552 & 0.0555 \\ 0.1560 & 1.3260 & 0.7760 & -0.0312 & -0.2652 & -0.1552 \\ 3.8800 & 2.0800 & 4.6800 & -0.7760 & -0.4160 & -0.9360 \\ 4.6800 & 3.8800 & 2.0800 & -0.9360 & -0.7760 & -0.4160 \\ 2.0800 & 4.6800 & 3.8800 & -0.4160 & -0.9360 & -0.7760 \end{bmatrix}$$

$A_3 =$

$$\begin{bmatrix} 0.7760 & 1.7160 & -0.6240 & -0.1552 & -0.3432 & 0.1248 \\ 0.0693 & 0.7760 & 1.6293 & -0.0139 & -0.1552 & -0.3259 \\ 1.1960 & 0.0260 & 0.7760 & -0.2392 & -0.0052 & -0.1552 \\ 3.8800 & 4.6800 & 2.0800 & -0.7760 & -0.9360 & -0.4160 \\ 2.0800 & 3.8800 & 4.6800 & -0.4160 & -0.7760 & -0.9360 \\ 4.6800 & 2.0800 & 3.8800 & -0.9360 & -0.4160 & -0.7760 \end{bmatrix}$$

$A_4 =$

$$\begin{bmatrix} 0.1000 & 1.9500 & -2.6000 & -0.0200 & -0.3900 & 0.5200 \\ -0.8667 & 0.1000 & 1.7333 & 0.1733 & -0.0200 & -0.3467 \\ 0.6500 & -0.9750 & 0.1000 & -0.1300 & 0.1950 & -0.0200 \\ 0.5000 & 0.0000 & 0.0000 & -0.1000 & 0.0000 & 0.0000 \\ 0.0000 & 0.5000 & 0.0000 & 0.0000 & -0.1000 & 0.0000 \\ 0.0000 & 0.0000 & 0.5000 & 0.0000 & 0.0000 & -0.1000 \end{bmatrix}$$

$A_5 =$

$$\begin{bmatrix} 0.7760 & 1.5860 & -0.6240 & -0.1552 & -0.3172 & 0.1248 \\ 0.4160 & 0.7760 & 1.4560 & -0.0832 & -0.1552 & -0.2912 \\ 0.8060 & 0.3510 & 0.7760 & -0.1612 & -0.0702 & -0.1552 \\ 3.8800 & 2.0800 & 4.6800 & -0.7760 & -0.4160 & -0.9360 \\ 4.6800 & 3.8800 & 2.0800 & -0.9360 & -0.7760 & -0.4160 \\ 2.0800 & 4.6800 & 3.8800 & -0.4160 & -0.9360 & -0.7760 \end{bmatrix}$$

$A_6 =$

$$\begin{bmatrix} 0.1000 & -1.9500 & 2.6000 & -0.0200 & 0.3900 & -0.5200 \\ 0.8667 & 0.1000 & -1.7333 & -0.1733 & -0.0200 & 0.3467 \\ -0.6500 & 0.9750 & 0.1000 & 0.1300 & -0.1950 & -0.0200 \\ 0.5000 & 0.0000 & 0.0000 & -0.1000 & 0.0000 & 0.0000 \\ 0.0000 & 0.5000 & 0.0000 & 0.0000 & -0.1000 & 0.0000 \\ 0.0000 & 0.0000 & 0.5000 & 0.0000 & 0.0000 & -0.1000 \end{bmatrix}$$

$A_7 =$

$$\begin{bmatrix} 0.7760 & -0.2340 & 1.9760 & -0.1552 & 0.0468 & -0.3952 \\ 0.9360 & 0.7760 & -0.1040 & -0.1872 & -0.1552 & 0.0208 \\ 0.5460 & 1.0010 & 0.7760 & -0.1092 & -0.2002 & -0.1552 \\ 3.8800 & 4.6800 & 2.0800 & -0.7760 & -0.9360 & -0.4160 \\ 2.0800 & 3.8800 & 4.6800 & -0.4160 & -0.7760 & -0.9360 \\ 4.6800 & 2.0800 & 3.8800 & -0.9360 & -0.4160 & -0.7760 \end{bmatrix}$$

$A_8 =$

$$\begin{bmatrix} 0.7760 & 3.5360 & -3.2240 & -0.1552 & -0.7072 & 0.6448 \\ -0.4507 & 0.7760 & 3.1893 & 0.0901 & -0.1552 & -0.6379 \\ 1.4560 & -0.6240 & 0.7760 & -0.2912 & 0.1248 & -0.1552 \\ 3.8800 & 2.0800 & 4.6800 & -0.7760 & -0.4160 & -0.9360 \\ 4.6800 & 3.8800 & 2.0800 & -0.9360 & -0.7760 & -0.4160 \\ 2.0800 & 4.6800 & 3.8800 & -0.4160 & -0.9360 & -0.7760 \end{bmatrix}$$

$A_9 =$

$$\begin{bmatrix} 0.7760 & -2.1840 & 4.5760 & -0.1552 & 0.4368 & -0.9152 \\ 1.8027 & 0.7760 & -1.8373 & -0.3605 & -0.1552 & 0.3675 \\ -0.1040 & 1.9760 & 0.7760 & 0.0208 & -0.3952 & -0.1552 \\ 3.8800 & 4.6800 & 2.0800 & -0.7760 & -0.9360 & -0.4160 \\ 2.0800 & 3.8800 & 4.6800 & -0.4160 & -0.7760 & -0.9360 \\ 4.6800 & 2.0800 & 3.8800 & -0.9360 & -0.4160 & -0.7760 \end{bmatrix}$$

$$B = \left[\begin{array}{ccc} 0.1000 & 0 & 0 \\ 0 & 0.0667 & 0 \\ 0 & 0 & 0.0500 \\ \hline \multicolumn{3}{c}{0_3} \end{array}\right]$$

Acknowledgments

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